ANISOTROPIC ELLIPTIC EQUATIONS WITH GENERAL GROWTH IN THE GRADIENT AND HARDY-TYPE POTENTIALS

FRANCESCO DELLA PIETRA AND NUNZIA GAVITONE

Abstract. In this paper we give existence and regularity results for the solutions of problems whose prototype is

$$\begin{cases} -\mathcal{Q}v = \beta(|v|)H(Dv)^q + \dfrac{\lambda}{H^o(x)^p}|v|^{p-2}v + f(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

with Ω bounded domain of $\mathbb{R}^N$, $N \geq 2$, $0 < p-1 < q \leq p < N$, β is a nonnegative continuous function and $\lambda \geq 0$. Moreover, H is a general norm of $\mathbb{R}^N$, H^o is its polar and $\mathcal{Q}v := \sum_{i=1}^N \frac{\partial}{\partial x_i}(H(Dv)^{p-1}H_{\xi_i}(Dv))$.

1. Introduction

In the present paper we study existence and regularity results for Dirichlet problems which involve a class of nonlinear elliptic operators in divergence form, under the influence of lower-order terms. Given a function $H\colon \mathbb{R}^N \to [0,+\infty[$, $N \geq 2$, convex, 1-homogeneous and in $C^1(\mathbb{R}^N \setminus \{0\})$, we deal with operators whose prototype is the following:

$$\mathcal{Q}v := \sum_{i=1}^{N} \frac{\partial}{\partial x_i}(H(Dv)^{p-1}H_{\xi_i}(Dv)), \tag{1.1}$$

with $1 < p < N$. In general, $\mathcal{Q}$ is highly nonlinear, and extends some well-known classes of operators. In particular, for $H(\xi) = (\sum_k |\xi_k|^r)^{\frac{1}{r}}$, $r > 1$, $\mathcal{Q}$ becomes

$$\mathcal{Q}v = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left(\sum_{k=1}^{N} \left| \frac{\partial v}{\partial x_k} \right|^r \right)^{(p-r)/r} \left| \frac{\partial v}{\partial x_i} \right|^{r-2} \frac{\partial v}{\partial x_i} \right).$$

Note that for $r = 2$, it coincides with the usual p-Laplace operator, while for $r = p$ it is the so-called pseudo-p-Laplace operator.

This kind of operator has been studied in several papers (see for instance [5], [17], [21], [20], [22] for $p = 2$, and [8], [9], [30] for $1 < p < \infty$).

The aim of this paper is to study a class of equations whose prototype involves in its principal part the operator (1.1), and a Hardy-type potential. Moreover, we are also interested in the influence of a lower-order term depending on the gradient. The problems we deal with are modeled on the following:

$$\begin{cases} -\mathcal{Q}v = \beta(|v|)H(Dv)^q + \dfrac{\lambda}{H^o(x)^p}|v|^{p-2}v + f(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

Date: September 6, 2012.

1991 *Mathematics Subject Classification.* 35J65,35B45.

Key words and phrases. Nonlinear elliptic boundary value problems, Hardy inequalities, a priori estimates, convex symmetrization.

with Ω bounded domain of $\mathbb{R}^N$, with $0 \in \Omega$, $N \geq 2$, $1 < p < N$, $p-1 < q \leq p$, β is a nonnegative continuous function, $\lambda \geq 0$ and f a measurable function on whose summability we will make different assumptions. Moreover, we denote with H^o the polar function of H (see Section 2 for the precise definition).

When $H(\xi) = |\xi|$, the general problem (1.2) reduces to

$$\begin{cases} -\Delta_p v = \beta(|v|)|Dv|^q + \dfrac{\lambda}{|x|^p}|v|^{p-2}v + f(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Equations like (1.3) have been widely studied in literature either in the case $\lambda = 0$ or when $\beta = 0$.

In the case $\lambda = 0$, it is well-known that for a general continuous function β, a smallness assumption on some norm of f is needed in order to have existence results (see, for example, [23–25, 27, 28] for $\beta \equiv 1$, or [1, 19, 29, 33, 39] in the general case). Moreover, under some appropriate hypotheses on the function β, it is possible to remove the smallness condition of f (see [16, 35]).

In the case $\beta = 0$, the existence of a solution of (1.3) can be proved under the assumption of $\lambda \leq \Lambda_N$ (see [26]), where Λ_N denotes the best constant in the classical Hardy inequality. Moreover, if $p = 2$ in [15] some regularity results are proved. Surprisingly, the regularity of the solutions also depend on the size of λ.

As matter of fact, the influence of both terms in the right-hand side of (1.3) has been studied in [2, 3] in the case β is a positive constant. In such papers, some existence and nonexistence results are proved. In particular, it is shown that when $p = q$ there is no positive solution, even in a very weak sense, when $f > 0$ and $\lambda > 0$.

Recently, in [31] the authors study problems whose model is (1.3) with $q = p$ and β nonconstant, giving some existence and regularity results. More precisely, they prove that under a structural assumption on β, which involves its behavior at infinity, if $f \in L^m(\Omega)$, $m > 1$ there exists a solution of (1.3) whose regularity depends on m and on the size of λ.

As regards the general problem (1.2), in [20] we investigated the particular case with $\beta \equiv 0$ and $p = 2$, namely

$$\begin{cases} -\mathcal{Q}v = \dfrac{\lambda}{H^o(x)^2}v + f(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Here Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 3$, containing the origin, and λ is a nonnegative constant. We studied the existence and the regularity of the solutions of (1.4) with respect to the summability of f, chosed in suitable Lorentz spaces, and the size of λ.

Our purpose is to study problem (1.2) for a general $\beta \geq 0$ and $p-1 < q \leq p$. In particular, the novelties of the paper relies in two main topics. First, using symmetrization techniques we are able to fully analyse the case $q < p$ that, up to our knowledge, also in the Euclidean case has been studied only in particular cases. Second, taking into account the structure of the operator, we use a suitable symmetrization argument, involving the so-called convex symmetrization (see [5], and Section 2 for the definition), which allows to obtain optimal results (see Remark 3.4).

To study problem (1.2), we investigate the existence and regularity issues by choosing f in appropriate Lorentz spaces. Under suitable assumptions on β, we find a critical value of λ, which depends on β and on the summability of f, such that a solution of (1.2) exists.

Moreover, we prove that the obtained solution and its gradient belong to suitable Lorentz spaces (see Section 3, Theorems 3.1, 3.2).

As usual, a key role is played by uniform estimates of the solutions of appropriate approximating problems (see Section 4), obtained by means of the quoted convex symmetrization.

For ease of reading, we state the main results in Section 3, adding some comments and remarks. Their proofs are contained in sections 4 and 5.

2. Notation and preliminaries

Let $N \geq 2$, and $H : \mathbb{R}^N \to [0, +\infty[$ be a $C^1(\mathbb{R}^N \setminus \{0\})$ function such that

$$H(t\xi) = |t|H(\xi), \quad \forall \xi \in \mathbb{R}^N, \ \forall t \in \mathbb{R}. \tag{2.1}$$

Moreover, suppose that there exist two positive constants $c_1 \leq c_2$ such that

$$c_1|\xi| \leq H(\xi) \leq c_2|\xi|, \quad \forall \xi \in \mathbb{R}^N. \tag{2.2}$$

The polar function $H^o : \mathbb{R}^N \to [0, +\infty[$ of H is defined as

$$H^o(v) := \sup_{\xi \neq 0} \frac{\xi \cdot v}{H(\xi)}.$$

It is easy to verify that also H^o is a convex function which satisfies properties (2.1) and (2.2). Furthermore,

$$H(v) = \sup_{\xi \neq 0} \frac{\xi \cdot v}{H^o(\xi)}.$$

The set

$$\mathcal{W} = \{\xi \in \mathbb{R}^N : H^o(\xi) < 1\}.$$

is the so-called Wulff shape centered at the origin. We put $\kappa_N = |\mathcal{W}|$, and denote $\mathcal{W}_r = r\mathcal{W}$.

In the following, we often make use of some well-known properties of H and H^o:

$$H(DH^o(\xi)) = H^o(DH(\xi)) = 1, \quad \forall \xi \in \mathbb{R}^N \setminus \{0\},$$
$$H^o(\xi) DH(DH^o(\xi)) = H(\xi) DH^o(DH(\xi)) = \xi, \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

Let Ω be an open subset of $\mathbb{R}^N$. The total variation of a function $u \in BV(\Omega)$ with respect to H is (see [7]):

$$\int_\Omega |Du|_H = \sup \left\{ \int_\Omega u \operatorname{div} \sigma dx : \sigma \in C_0^1(\Omega; \mathbb{R}^N), \ H^o(\sigma) \leq 1 \right\}.$$

This yields the following definition of anisotropic perimeter of $F \subset \mathbb{R}^N$ in Ω:

$$P_H(F; \Omega) = \int_\Omega |D\chi_F|_H = \sup \left\{ \int_F \operatorname{div} \sigma dx : \sigma \in C_0^1(\Omega; \mathbb{R}^N), \ H^o(\sigma) \leq 1 \right\}.$$

The following co-area formula for the anisotropic perimeter

$$\int_{\{u>t\}} H(Du) dx = \int_\Omega P_H(\{u > s\}, \Omega) \, ds, \quad \forall u \in BV(\Omega) \tag{2.3}$$

holds, moreover

$$P_H(F; \Omega) = \int_{\Omega \cap \partial^* F} H(\nu_F) d\mathcal{H}^{N-1}$$

where $\mathcal{H}^{N-1}$ is the $(N-1)$–dimensional Hausdorff measure in $\mathbb{R}^N$, $\partial^* F$ is the reduced boundary of F and ν_F is the outer normal to F (see [7]).

The anisotropic perimeter of a set F is finite if and only if the usual Euclidean perimeter

$$P(F;\Omega)=\sup\left\{\int_F \operatorname{div}\sigma dx\colon \sigma\in C_0^1(\Omega;\mathbb{R}^N),\ |\sigma|\le 1\right\}.$$

is finite. Indeed, by properties (2.1) and (2.2) we have that

$$\frac{1}{c_2}|\xi|\le H^o(\xi)\le\frac{1}{c_1}|\xi|, \tag{2.4}$$

and then

$$c_1P(E;\Omega)\le P_H(E;\Omega)\le c_2P(E;\Omega).$$

A fundamental inequality for the anisotropic perimeter is the isoperimetric inequality

$$P_H(E;\mathbb{R}^N)\ge N\kappa_N^{\frac{1}{N}}|E|^{1-\frac{1}{N}}, \tag{2.5}$$

which holds for any measurable subset E of $\mathbb{R}^N$ (see for instance [5]).

Finally, if $u\in W^{1,1}(\Omega)$, then (see [7])

$$\int_\Omega |Du|_H=\int_\Omega H(Du)dx.$$

2.1. **Rearrangements, convex symmetrization and Lorentz spaces.** We recall some basic definition on rearrangements. Let Ω be an bounded open set of $\mathbb{R}^N$, $u:\Omega\to\mathbb{R}$ be a measurable function, and denote with $|\Omega|$ the Lebesgue measure of Ω.

The distribution function of u is the map $\mu_u:\mathbb{R}\to[0,\infty[$ defined by

$$\mu_u(t)\ =\ |\{x\in\Omega:|u(x)|>t\}|.$$

Such function is decreasing and right continuous.

The decreasing rearrangement of u is the map $u^*:\ [0,\infty[\to\mathbb{R}$ defined by

$$u^*(s):=\sup\{t\in\mathbb{R}:\mu_u(t)>s\}.$$

The function u^* is the generalized inverse of μ_u.

Following [5], the convex symmetrization of u is the function $u^\star(x)$, $x\in\Omega^\star$ defined by:

$$u^\star(x)=u^*(\kappa_N H^o(x)^N),$$

where $\Omega^\star$ is a set homothetic to the Wulff shape centered at the origin having the same measure of Ω, that is, $\Omega^*=\mathcal{W}_R$, with $R=(\frac{|\Omega|}{\kappa_N})^{1/N}$.

The inequalities stated below will be useful in the next sections.

Proposition 2.1. *Suppose $\lambda>0$, $1\le\gamma<+\infty$. Let ψ a nonnegative measurable function on $(0,+\infty)$. The following inequalities hold:*

$$\int_0^{+\infty}\left(t^\lambda\int_t^{+\infty}\psi(s)ds\right)^\gamma\frac{dt}{t}\le\lambda^{-\gamma}\int_0^{+\infty}(t^{1+\lambda}\psi(t))^\gamma\frac{dt}{t} \tag{2.6}$$

and

$$\int_0^{+\infty}\left(t^{-\lambda}\int_0^{t}\psi(s)ds\right)^\gamma\frac{dt}{t}\le\lambda^{-\gamma}\int_0^{+\infty}(t^{1-\lambda}\psi(t))^\gamma\frac{dt}{t}. \tag{2.7}$$

We recall that a measurable function $u : \Omega \to \mathbb{R}$ belongs to the Lorentz space $L(p,q)$, $1 < p < +\infty$, if the quantity

$$\|u\|_{p,q} = \begin{cases} \left\{ \displaystyle\int_0^{+\infty} \left[t^{1/p} u^*(t) \right]^q \frac{dt}{t} \right\}^{1/q}, & 1 \le q < +\infty, \\ \displaystyle\sup_{0<t<+\infty} t^{1/p} u^*(t), & q = +\infty, \end{cases}$$

is finite.

In general $\|u\|_{p,q}$ is not a norm. As matter of fact, it is possible to introduce a metric in $L(p,q)$ in the following way. Let us define

$$\|u\|_{(p,q)} = \|u^{**}\|_{p,q},$$

with $u^{**}(t) = t^{-1} \int_0^t u^*(\sigma)\, d\sigma$. We observe that also u^{**} is a decreasing function, hence $(u^{**})^* = u^{**}$. By means of the inequality (2.7) and the properties of rearrangements, we have that for $1 < p < +\infty$ and $1 \le q \le +\infty$,

$$\|u\|_{p,q} \le \|u\|_{(p,q)} \le \frac{p}{p-1} \|u\|_{p,q}.$$

Hence, the topology induced by $\|\cdot\|_{(p,q)}$ and $\|\cdot\|_{p,q}$ is the same, that is

$$u_n \to u \text{ in } L(p,q) \iff \lim_n \|u_n - u\|_{p,q} = 0.$$

We stress that, for any fixed p, the Lorentz spaces $L(p,q)$ increase as the secondary exponent q increases. Indeed, if $1 \le q \le r \le +\infty$, there exists a constant $C > 0$ depending only on p, q and r such that

$$\|u\|_{p,r} \le C \|u\|_{p,q}.$$

More generally, the $L(p,q)$ spaces are related in the following way:

$$L^r \subset L(p,1) \subset L(p,q) \subset L(p,p) = L^p \subset L(p,r) \subset L(p,\infty) \subset L^q,$$

for $1 < q < p < r < +\infty$.

More details on Lorentz spaces can be found, for example, in [11].

In the next sections, a basic tool will be the Hardy inequality, stated below.

Proposition 2.2. *For any $u \in W^{1,p}(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} H(Du)^p dx \ge \Lambda_N \int_{\mathbb{R}^N} \frac{|u|^p}{H^o(x)^p} dx, \tag{2.8}$$

and the constant $\Lambda_N = \left(\frac{N-p}{p}\right)^p$ is optimal, and not achieved.

If $H(\xi) = |\xi|$, (2.8) is the classical Hardy inequality. For a general H, (2.8) is proved in [40].

Remark 2.1. The inequality (2.8), using the Pólya Szegö inequality in the anisotropic case (see [5]), can be rewritten as

$$\|u\|_{p^*,p} \le \frac{p}{(N-p)\kappa_N^{1/N}} \left(\int_{\mathbb{R}^N} H(Du)^p dx \right)^{1/p},$$

recovering the well-known result $W_0^{1,p}(\Omega) \subset L(p^*, p)$ (see also [4]).

3. Statement of the problem and main results

In this section we state the problem and the main results of the paper. The proofs of the theorems are contained in sections 4 and 5.

Our aim is to prove a priori estimates and existence results for problems of the type

$$\begin{cases} -\operatorname{div}(a(x,u,Du)) = b(x,u,Du) + g(x,u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $a\colon \Omega\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function verifying

$$a(x,s,\xi)\cdot\xi \geq H(\xi)^p, \tag{3.2}$$

with $1<p<N$, and

$$|a(x,s,\xi)| \leq \alpha(|\xi|^{p-1} + |s|^{p-1} + k(x)), \tag{3.3}$$

for a.e. $x\in\Omega$, for any $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $\alpha>0$ and $k\in L_+^{p'}(\Omega)$. Moreover,

$$(a(x,s,\xi) - a(x,s,\xi'))\cdot(\xi-\xi') > 0, \tag{3.4}$$

for a.e. $x\in\Omega$, for all $s\in\mathbb{R}, \xi\neq\xi'\in\mathbb{R}^N$. As regards the lower order terms, we suppose that $b\colon \Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ and $g\colon\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions such that

$$|b(x,s,\xi)| \leq \beta(|s|)H(\xi)^q \tag{3.5}$$

for a.e. $x\in\Omega$, for any $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, with $p-1<q\leq p$, and $\beta\colon[0,+\infty[\to[0,+\infty[$ is continuous. Moreover,

$$\begin{gathered} g(x,s)s \leq c(x)|s|^p, \\ |g(x,s)| \leq d(x)|s|^{p-1}, \end{gathered} \tag{3.6}$$

for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$, where $c(x)$ and $d(x)$ are measurable functions in Ω such that

$$(c^+)^\star(x) \leq \frac{\lambda}{H^o(x)^p}, \quad \forall x\in\Omega^\star, \tag{3.7}$$

with $\lambda\geq 0$, and $d(x)\in L\left(\frac{N}{p},\infty\right)$.

Finally, we take f is in some suitable Lebesgue or Lorentz spaces which will be specified in the following.

If $f\in L((p^*)',p')$, we say that $u\in W_0^{1,p}(\Omega)$ is a weak solution of (3.1) if

$$\int_\Omega a(x,u,Du)\cdot D\varphi\,dx = \int_\Omega [b(x,u,Du)+g(x,u)+f]\varphi\,dx,$$

for any $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

The summability condition on f given above is the one which yields solutions in the energy space $W_0^{1,p}(\Omega)$.

In order to state the main results, we need further assumptions on β and λ. First of all, let

$$B(\infty) = \left(\frac{|\Omega|}{\kappa_N}\right)^{\frac{p-q}{N}} \left(\int_0^\infty \beta(t)^{\frac{1}{q-(p-1)}}\,dt\right)^{q-(p-1)} < \infty, \tag{3.8}$$

and, for $1<m<\frac{N}{p}$, define the value $\lambda(m)$ as

$$\lambda(m) = e^{-B(\infty)}\frac{N(m-1)(N-mp)^{p-1}}{m^p(p-1)^{p-1}}. \tag{3.9}$$

The first result we get is the following.

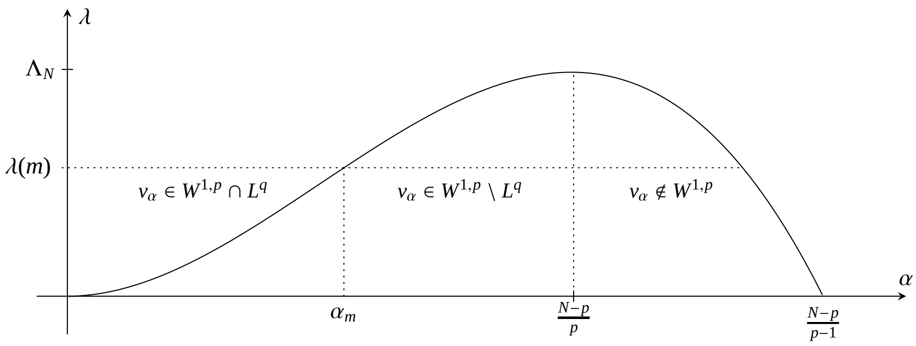

FIGURE 1. Graph of the function $F(\alpha) = -(p-1)\alpha^p + (N-p)\alpha^{p-1}$ in Remark 3.1. Here we consider the case $\frac{N}{p} > m > (p^*)'$, and $q = \frac{Nm(p-1)}{N-mp}$.

Theorem 3.1. *Suppose that* (3.2) ÷ (3.7), (3.8) *hold. Moreover, suppose that* $0 \le \lambda < \Lambda_N e^{-B(\infty)}$. *The following results hold:*

(i) *if* $f \in L((p^*)', p')$, *problem* (3.1) *admits a weak solution* $u \in W_0^{1,p}(\Omega)$;

(ii) *if* $f \in L(m, \sigma)$, *with* $(p^*)' < m < \frac{N}{p}$, $\max\left\{\frac{1}{p-1}, 1\right\} \le \sigma \le +\infty$, *and* $0 \le \lambda < \lambda(m)$, *then there exists a weak solution* u *to* (3.1) *such that*

$$\|u\|_{\frac{Nm(p-1)}{N-mp}, \sigma(p-1)} \le C\|f\|_{m,\sigma}^{\frac{1}{p-1}}.$$

From the embedding of Lorentz spaces, the above theorem gives immediately the following result.

Corollary 3.1. *Suppose that the hypotheses* (3.2) – (3.7),(3.8) *hold. If* $f \in L^m(\Omega)$, *with* $(p^*)' < m < \frac{N}{p}$, *and* $0 \le \lambda < \lambda(m)$, *with* $\lambda(m)$ *as in* (3.9), *then there exists a weak solution* u *to* (3.1) *such that*

$$\|u\|_{\frac{Nm(p-1)}{N-mp}} \le C,$$

for some constant C *depending on the norm* $\|f\|_m$.

Remark 3.1. At least in the case $\beta = 0$, the value $\lambda = \lambda(m)$, with $\lambda(m)$ as in (3.9), is optimal in order to obtain the estimates in (4.2). Let $(p^*)' = \frac{Np}{Np-N+p} < m < \frac{N}{p}$, and $0 < \lambda < \Lambda_N = \left(\frac{N-p}{p}\right)^p$. For sake of simplicity, we prove the optimality of $\lambda(m)$ in the case of estimates in Lebesgue spaces, that is when $\sigma = \frac{Nm}{N-mp}$.

We first consider radial solutions $v(x) = v(r)$, $r = H^o(x)$, $x \in \mathcal{W}$, of the equation

$$-\mathcal{Q}v = \frac{\lambda}{H^o(x)^p}|v|^{p-2}v, \quad x \in \mathcal{W},$$

where $\mathcal{W}$ is the Wulff shape. We also suppose that $H \in C^2(\mathbb{R}^N \setminus \{0\})$. In particular, we look for solutions $v = v_\alpha = r^{-\alpha}$, with $\alpha > 0$, which satisfy the ODE

$$-|v'|^{p-2}\left((p-1)v'' + \frac{N-1}{r}v'\right) = \frac{\lambda}{r^p}|v|^{p-2}v \quad \text{in }]0,1[. \tag{3.10}$$

Then, v_α solves (3.10) if α satisfies the equation

$$F(\alpha) = \lambda, \quad \text{where } F(\alpha) := -(p-1)\alpha^p + (N-p)\alpha^{p-1}.$$

We stress that $v_\alpha \in L^{\frac{Nm(p-1)}{N-mp}}(\mathcal{W})$ if $\alpha < \alpha_m = \frac{N-mp}{m(p-1)}$. Moreover, $\alpha_m < \frac{N-p}{p}$, being $m > (p^*)'$. Hence $v \in W^{1,p}(\mathcal{W})$. Furthermore, $F(\alpha_m) = \lambda(m)$. In order to prove the optimality of $\lambda(m)$, we observe that the positive function $z_\alpha(x) = z_\alpha(r) = v_\alpha(r) - 1$ is such that

$$\begin{cases} -\mathcal{Q}z_\alpha = \dfrac{\lambda(m)}{H^o(x)^p} z_\alpha^{p-1} + g(H^o(x)) & \text{in } \mathcal{W}, \\ z_\alpha = 0 & \text{on } \partial\mathcal{W}, \end{cases}$$

with

$$g(r) = \lambda(m)\frac{(z_\alpha+1)^{p-1} - z_\alpha^{p-1}}{r^p} = \lambda(m)\frac{1-(1-r^{\alpha_m})^{p-1}}{r^{\alpha_m}} \cdot \frac{1}{r^{p+\alpha_m(p-2)}}.$$

The condition $m < \frac{N}{p}$ gives that $g \in L^m(\mathcal{W})$. Nevertheless, for $\alpha \geq \alpha_m$, z_α does not belong to $L^{\frac{Nm(p-1)}{N-mp}}(\mathcal{W})$. This prove the optimality of $\lambda(m)$.

Next step is to state an existence and regularity result for problems whose datum f is in L^m, $m > 1$. To this aim, we deal with entropy solutions.

Following [10], for a general $f \in L^1(\Omega)$ we will say that a measurable function u is an entropy solution of (3.1) if $g(x,u), b(x,u,Du) \in L^1(\Omega)$ and, for any $k > 0$, $T_k(u) \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega a(x,u,Du)\cdot DT_k(u-\varphi)\,dx \leq \int_\Omega [b(x,u,Du) + g(x,u) + f]T_k(u-\varphi)\,dx, \tag{3.11}$$

for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. When $T_k(u) \in W_0^{1,p}(\Omega)$ for any $k > 0$, it is possible to define the weak gradient of u, namely Du, as the function such that $DT_k(u) = (Du)\chi_{\{|u|\leq k\}}$, for any $k > 0$ (see [10]).

The following result holds.

Theorem 3.2. *Let us suppose that* (3.2) – (3.7), (3.8) *hold. If* $f \in L(m,\sigma)$, $1 < m < (p^*)'$, $p' \leq \sigma \leq \infty$, *and* $0 \leq \lambda < \lambda(m)$, *with* $\lambda(m)$ *as in* (3.9), *then there exists an entropy solution of* (3.1) *such that*

$$\left\| H(Du)^{p-1} \right\|_{\frac{Nm}{N-m},\sigma} \leq C\|f\|_{m,\sigma}. \tag{3.12}$$

Remark 3.2. We stress that the assumptions of Theorem 3.2 do not allow to obtain the estimate (3.12) for $\sigma = m$. As matter of fact, defined $\bar{m} = \frac{N}{Np-N+1}$, when $f \in L^m(\Omega)$, $\max\{1,\bar{m}\} \leq m < (p^*)'$, it is possible to prove the existence of a solution such that

$$\|H(Du)\|_{\frac{Nm(p-1)}{N-m}} \leq C(\|f\|_m).$$

We refer the reader to remarks 4.1 and 5.1.

In the case $1 < m < \max\{1,\bar{m}\}$, the solutions we obtain no longer belong to a Sobolev space, but Theorem 3.2 guarantees that there exists a solution u such that, for example,

$$\left\| H(Du)^{p-1} \right\|_{\frac{Nm}{N-m},\infty} \leq C\|f\|_{m,\infty}.$$

Actually, $\bar{m} > 1$ only if $p < 2 - \frac{1}{N}$. In this case an estimate of the type

$$\|H(Du)^{p-1}\|_{\frac{Nm}{N-m}} \leq C(\|f\|_m)$$

holds for any $1 < m \leq \bar{m}$ (see Remark 4.1).

Remark 3.3. We explictly observe that, in general, the above Theorem does not hold for $f \in L^1(\Omega)$. For example, it has been proved in [15] that, when $H(\xi) = |\xi|$, $p = 2$ and $\beta \equiv 0$, for any $\lambda > 0$ no a priori estimate holds for problem (3.1). As matter of fact, when $\lambda = 0$, if $\beta \in L^1(\Omega)$ and $p = q$ it is possible to prove the existence of a solution of (3.1) (see for example [36], [34]).

Remark 3.4. We stress that the bounds (2.2) and (2.4) on H and H^o, and the conditions (3.2), (3.5) and (3.7) give that

$$a(x,s,\xi)\cdot\xi \ge c_1^p|\xi|^p, \quad |b(x,s,\xi)| \le c_2^q\beta(|s|)|\xi|^q,$$

and

$$(c^+)^\star(x) \le \frac{\lambda c_2^p}{|x|^p}.$$

Hence, under the above growth conditions, the classical Schwarz symmetrization tecnique can be applied to problem (3.1). In this way, it is possible to obtain analogous results than Theorem 4.2, and consequently Theorem 3.1, but requiring a stronger assumption on the smallness of $\lambda > 0$ (see also [5] and Remark 4.1 in [20]). This justifies the use of the more general convex symmetrization.

Remark 3.5. Let $H(\xi) = |\xi|$. If $q = p$, the regularity estimates obtained in Theorems 4.2 and 4.3 are slightly more general than the analogous one contained in [31]. Indeed, in such paper the datum f in suitable Lebesgue space is considered, while we give optimal regularity results in Lorentz spaces.

4. A priori estimates and approximating problems

The first aim of this section is to prove three integro-differential inequalities for the rearrangements of solutions of (3.1), in the spirit of the comparison results contained, for instance, in [37], [38], [6], [23]. To prove such inequalities we need the additional assumption (3.8).

Theorem 4.1. *Suppose that* (3.2), (3.3), (3.5) ÷ (3.7) *hold, and* $f \in L((p^*)', p')$*. Moreover, suppose that* $\beta(s)$ *verifies* (3.8)*. Then any weak solution* $u \in W_0^{1,p}(\Omega)$ *of problem* (3.1) *satisfies*

$$-\frac{d}{dt}\int_{\{|u|>t\}} H(Du)^p dx \le e^{B(\infty)}\int_0^{\mu_u(t)} [(c^+)^*(s)u^*(s)^{p-1} + f^*(s)]ds, \quad \text{a.e. } t>0, \tag{4.1}$$

$$u^*(s) \le e^{\frac{B(\infty)}{p-1}}\left(N\kappa_N^{1/N}\right)^{-p'}\int_s^{|\Omega|} t^{-\frac{p'}{N'}}\left(\int_0^t [(c^+)^*(r)(u^*(r))^{p-1} + f^*(r)]dr\right)^{\frac{1}{p-1}} dt, \quad s\in]0,|\Omega|]. \tag{4.2}$$

Moreover, for any $\alpha > \frac{p'}{N'} - 1$ *we have that*

$$\begin{aligned}[H(Du)^*(s)]^p \le{}& \\ \le{}& e^{\frac{B(\infty)}{p-1}}\left(N\kappa_N^{1/N}\right)^{-p'}\left[\frac{1}{s^{\alpha+1}}\int_0^s t^{\alpha-\frac{p'}{N'}}\left(\int_0^t [(c^+)^*(r)(u^*(r))^{p-1} + f^*(r)]dr\right)^{p'} dt + \right.\\ &\left. +\frac{1}{s}\int_s^{|\Omega|} t^{-\frac{p'}{N'}}\left(\int_0^t [(c^+)^*(r)(u^*(r))^{p-1} + f^*(r)]dr\right)^{p'} dt\right], \quad s\in]0,|\Omega|[.\end{aligned} \tag{4.3}$$

Proof. Let $u \in W_0^{1,p}(\Omega)$ be a solution to (3.1). Using the following test function $\varphi \in W_0^{1,p}(\Omega)$,

$$\varphi(x) = \begin{cases} 0 & |u| \le t, \\ (|u| - t)\operatorname{sign} u & t < |u| \le t + h, \\ h \operatorname{sign} u & t + h < |u|, \end{cases}$$

by the hypotheses (3.2), (3.5) ÷ (3.7), and the Hardy-Littlewood inequality we obtain

$$-\frac{d}{dt}\int_{\{|u|>t\}} H(Du)^p\,dx \le \\ \le \int_{\{|u|>t\}} \beta(|u|)H(Du)^q\,dx + \int_0^{\mu_u(t)} \Big((c^+)^*(\sigma)u^*(\sigma)^{p-1} + f^*(\sigma)\Big)\,d\sigma.$$

By the continuity of β we have

$$\int_{\{|u|>t\}} \beta(|u|)H(Du)^q\,dx = \int_t^{+\infty} \beta(s)\left(-\frac{d}{ds}\int_{\{|u|>s\}} H(Du)^q\,dx\right)ds.$$

Hence, using also the Hölder inequality we get

$$\int_{|u|>t} \beta(|u|)H(Du)^q dx \le \int_t^{+\infty} \beta(s)\left[\left(-\frac{d}{ds}\int_{|u|>s} H(Du)^p dx\right)^{q/p} (-\mu_u'(s))^{1-q/p}\right] ds, \tag{4.4}$$

and

$$\left(-\frac{d}{ds}\int_{\{|u|>t\}} H(Du)^p dx\right)^{\frac{q}{p}} (-\mu_u'(s))^{1-\frac{q}{p}} \le \\ \le \left(-\frac{d}{ds}\int_{\{|u|>t\}} H(Du) dx\right)^{q-p} \left(-\frac{d}{ds}\int_{\{|u|>s\}} H(Du)^p dx\right)(-\mu_u'(s))^{p-q}.$$

The coarea formula (2.3) and the isoperimetric inequality (2.5) imply

$$\left(-\frac{d}{ds}\int_{\{|u|>s\}} H(Du)^p dx\right)^{\frac{q}{p}} (-\mu_u'(s))^{1-\frac{q}{p}} \le \\ \le \left(N\kappa_N^{1/N}\mu_u(s)^{1-\frac{1}{N}}\right)^{q-p} \left(-\frac{d}{ds}\int_{\{|u|>s\}} H(Du)^p dx\right)(-\mu_u'(s))^{p-q}. \tag{4.5}$$

So, from (4.4) and (4.5) we have

$$-\frac{d}{dt}\int_{\{|u|>t\}} H(Du)^p\,dx \le \\ \le \left(N\kappa_N^{1/N}\right)^{q-p} \int_t^{+\infty} \beta(s)\left(-\frac{d}{ds}\int_{\{|u|>s\}} H(Du)^p\,dx\right)\left(\frac{-\mu_u'(s)}{\mu_u(s)^{1-\frac{1}{N}}}\right)^{p-q} ds + \int_0^{\mu_u(t)} z(s)\,ds, \tag{4.6}$$

where for sake of brevity we set $z(s) = (c^+)^*(s)u^*(s)^{p-1} + f^*(s)$.

Now, using the Gronwall Lemma and the properties of rearrangements in (4.6), it follows that

(4.7)
$$-\frac{d}{dt}\int_{\{|u|>t\}} H(Du)^p\,dx \le \int_0^{\mu_u(t)} z(s)\exp\left\{\left(N\kappa_N^{1/N}\right)^{q-p}\int_t^{u^*(s)} \beta(y)\left(\frac{-\mu_u'(y)}{\mu_u(y)^{1-\frac{1}{N}}}\right)^{p-q} dy\right\} ds.$$

On the other hand, if $p-1<q<p$, using Hölder inequality we have

$$\int_t^{u^*(s)}\beta(y)\left(\frac{-\mu_u'(y)}{\mu_u(y)^{1-\frac{1}{N}}}\right)^{p-q}dy\le\left[\int_t^{u^*(s)}\beta(y)^{\frac{1}{1-p+q}}dy\right]^{1-p+q}\left[\int_t^{u^*(s)}\frac{-\mu_u'(y)}{\mu_u(y)^{1-\frac{1}{N}}}dy\right]^{p-q}. \tag{4.8}$$

(Observe that last inequality is trivial if $q=p$). Furthermore, by the properties of the distribution function μ of u, we have

$$\int_t^{u^*(s)}\frac{-\mu_u'(y)}{\mu_u(y)^{1-\frac{1}{N}}}dy\le\int_0^{+\infty}\frac{-\mu_u'(y)}{\mu_u(y)^{1-\frac{1}{N}}}dy\le N|\Omega|^{\frac{1}{N}}. \tag{4.9}$$

Using (4.8) and (4.9) in (4.7), we get

$$\begin{aligned}-\frac{d}{dt}\int_{\{|u|>t\}}H(Du)^p dx\le\int_0^{\mu(t)}z(s)\exp\left\{\left(\frac{|\Omega|}{\kappa_N}\right)^{\frac{p-q}{N}}\left[\int_t^{u^*(s)}(\beta(y))^{\frac{1}{1-p+q}}\,dy\right]^{1-p+q}\right\}ds\le\\ \le e^{B(\infty)}\int_0^{\mu_u(t)}[(c^+)^*(s)u^*(s)^{p-1}+f^*(s)]ds,\end{aligned} \tag{4.10}$$

where last inequality follows by (3.8), and $B(\infty)$ is finite by the assumption on β. This proves the inequality (4.1).

In order to show (4.2), using similarly as before the Hölder inequality, the coarea formula and the isoperimetric inequality in the left-hand side of (4.10), we get that

$$(-\mu_u'(t))^{-1}\le e^{\frac{1}{p-1}B(\infty)}\left(N\kappa_N^{1/N}\right)^{-p'}\mu_u(t)^{-\frac{p'}{N'}}\left(\int_0^{\mu_u(t)}[(c^+)^*(s)u^*(s)^{p-1}+f^*(s)]ds\right)^{1/(p-1)}.$$

Integrating between s and $|\Omega|$, we get (4.2).

Finally, following the argument contained in [6], we get last inequality (4.3). □

In order to get existence and regularity results for (3.1), we will consider the approximated problems

$$\begin{cases}-\operatorname{div}a(x,u_n,Du_n)=b_n(x,u_n,Du_n)+g_n(x,u_n)+f_n(x) & \text{in }\Omega,\\ u_n=0 & \text{on }\partial\Omega,\end{cases} \tag{4.11}$$

where $b_n(x,s,\xi)=T_n(b(x,s,\xi))$, $g_n(x,s)=T_n(g(x,s))$, $f_n(x)=T_n(f(x))$, and $T_n(s)=\max\{-n,\min\{s,n\}\}$ is the standard truncature function. Under the assumptions (3.2) ÷ (3.7), the existence of a weak solution $u_n\in W_0^{1,p}(\Omega)$ to problem (4.11) follows by the classical Leray-Lions result (see [32]). Moreover, such solution is bounded.

Now we use the inequalities proved in Theorem 4.1 in order to obtain some a priori estimates for problems (4.11). As stated in the introduction, an additional assumption on the smallness of the value λ, depending on the summability of f, is needed.

Theorem 4.2. *Suppose that the hypotheses* (3.2) ÷ (3.7), (3.8) *hold. Let $f\in L(m,\sigma)$, with $1<m<\frac{N}{p}$ and $\max\left\{\frac{1}{p-1},1\right\}\le\sigma\le+\infty$, and*

$$0\le\lambda<\lambda(m),$$

with $\lambda(m)$ as in (3.9)*. Then, the weak solutions u_n of* (4.11) *are such that*

$$\left\||u_n|^{p-1}\right\|_{\frac{Nm}{N-mp},\sigma}\le C\|f\|_{m,\sigma}, \tag{4.12}$$

for some positive constant C independent of n.

Proof. We first consider the case $\sigma = +\infty$. Problem (4.11) verifies the assumptions of Theorem 4.1. Hence, we can use the inequality (4.2) for u_n. Recalling that $(c^+)^*(\tau) \le \lambda \kappa_N^{\frac{p}{N}} \tau^{-\frac{p}{N}}$, $\tau \in]0, |\Omega|[$, we obtain that

$$u_n^*(s)^{p-1} \le e^{B(\infty)}(N\kappa_N^{\frac{1}{N}})^{-p} \left\{ \int_s^{|\Omega|} t^{\frac{-p'}{N'}} \left[\int_0^t \left(\lambda \kappa_N^{\frac{p}{N}} \tau^{-\frac{p}{N}} (u_n^*(\tau))^{p-1} + f_n^*(\tau) \right) d\tau \right]^{\frac{1}{p-1}} dt \right\}^{p-1} \le \tag{4.13}$$
$$\le e^{B(\infty)}(N\kappa_N^{\frac{1}{N}})^{-p} \left\{ \int_s^{|\Omega|} t^{\frac{-p'}{N'}} \left[\int_0^t \left(\lambda \kappa_N^{\frac{p}{N}} \|u_n^{p-1}\|_{\frac{Nm}{N-mp},\infty} + \|f_n\|_{m,\infty} \right) \tau^{-\frac{1}{m}} d\tau \right]^{\frac{1}{p-1}} dt \right\}^{p-1}.$$

Last inequality follows simply by the definition of the Lorentz norms.

Hence, from (4.13), recalling also that $|f_n| \le |f|$, we get

$$u_n^*(s)^{p-1} \le \left(\lambda e^{B(\infty)} N^{-p} \|u_n^{p-1}\|_{\frac{Nm}{N-mp},\infty} + C\|f\|_{m,\infty} \right) \left[\int_s^{|\Omega|} t^{-\frac{p'}{N'}} \left(\int_0^t \tau^{-\frac{1}{m}} d\tau \right)^{\frac{1}{p-1}} dt \right]^{p-1} \le$$
$$\le \left(\frac{\lambda}{\lambda(m)} \|u_n^{p-1}\|_{\frac{Nm}{N-mp},\infty} + C\|f\|_{m,\infty} \right) s^{-\frac{N-mp}{Nm}},$$

where C is a constant which does not depend on n. Finally, being $\lambda < \lambda(m)$, the above inequality gives that

$$\left\| |u_n|^{p-1} \right\|_{\frac{Nm}{N-mp},\infty} \le C\|f\|_{m,\infty},$$

and we get the thesis when $\sigma = \infty$. Now, suppose that $\max\left\{\frac{1}{p-1}, 1\right\} \le \sigma < +\infty$. For the sake of brevity, we denote with $z(\tau)$ the function $z(\tau) = (c^+)^*(\tau)(u_n^*(\tau))^{p-1} + f_n^*(\tau)$. As before, from (4.2) applied to u_n we obtain that

$$\left\| |u_n|^{p-1} \right\|_{\alpha,\sigma}^{\sigma} = \int_0^{|\Omega|} s^{\frac{\sigma}{\alpha}} u_n^*(s)^{\sigma(p-1)} \frac{ds}{s} \le \tag{4.14}$$
$$\le e^{\sigma B(\infty)} \left(N\kappa_N^{1/N} \right)^{-p\sigma} \int_0^{|\Omega|} \left(s^{\frac{1}{\alpha(p-1)}} \int_s^{|\Omega|} t^{-\frac{p'}{N'}} \left(\int_0^t z(\tau) d\tau \right)^{\frac{1}{p-1}} dt \right)^{\sigma(p-1)} \frac{ds}{s} \le$$
$$\le e^{\sigma B(\infty)} \left(N\kappa_N^{1/N} \right)^{-p\sigma} [\alpha(p-1)]^{\sigma(p-1)} \int_0^{|\Omega|} \left[s^{\frac{1}{\alpha}-1+\frac{p}{N}} \int_0^s z(\tau) d\tau \right]^{\sigma} \frac{ds}{s},$$

where last inequality is obtained by using (2.6), being $\sigma(p-1) \ge 1$.

Let us observe that

$$\frac{1}{\alpha} - 1 + \frac{p}{N} < 0 \iff \alpha > \frac{N}{N-p}. \tag{4.15}$$

If this is the case, being $\sigma \ge 1$, by (2.7) we get from (4.14) that

$$\left\| |u_n|^{p-1} \right\|_{\alpha,\sigma}^{\sigma} \le \tag{4.16}$$
$$\le e^{\sigma B(\infty)} (N\kappa_N^{1/N})^{-p\sigma} [\alpha(p-1)]^{\sigma(p-1)} \left(1 - \frac{p}{N} - \frac{1}{\alpha} \right)^{-\sigma} \int_0^{|\Omega|} \left[s^{\frac{1}{\alpha}+\frac{p}{N}} z(s) \right]^{\sigma} \frac{ds}{s} =$$
$$= K \int_0^{|\Omega|} \left[s^{\frac{1}{\alpha}+\frac{p}{N}} z(s) \right]^{\sigma} \frac{ds}{s}.$$

Hence, using the Minkowski inequality, we get that

$$\frac{1}{K^{\frac{1}{\sigma}}}\left\||u_n|^{p-1}\right\|_{\alpha,\sigma} \le \left(\int_0^{|\Omega|}\left[s^{\frac{1}{\alpha}+\frac{p}{N}}z(s)\right]^{\sigma}\frac{ds}{s}\right)^{\frac{1}{\sigma}} \le \left(\int_0^{|\Omega|}\left(s^{\frac{p}{N}}(c^+)^*(s)\right)^{\sigma}s^{\frac{\sigma}{\alpha}}u_n^*(s)^{\sigma(p-1)}\frac{ds}{s}\right)^{\frac{1}{\sigma}} + \left(\int_0^{|\Omega|}\left(s^{\frac{1}{\alpha}+\frac{p}{N}}f_n^*(s)\right)^{\sigma}\frac{ds}{s}\right)^{\frac{1}{\sigma}}. \tag{4.17}$$

Being $(c^+)^*(s) \le \lambda\kappa_N^{\frac{p}{N}}s^{-\frac{p}{N}}$, writing explicitly the value of K in (4.16), (4.17) implies

$$\kappa_N^{\frac{p}{N}}\left(\frac{\alpha(N-p)-N}{e^{B(\infty)}N^{1-p}\alpha^p(p-1)^{p-1}}-\lambda\right)\left\||u_n|^{p-1}\right\|_{\alpha,\sigma} \le \|f_n\|_{\frac{N\alpha}{N+\alpha p},\sigma}.$$

Hence, for $m = \frac{N\alpha}{N+\alpha p}$ we have that $\alpha = \frac{Nm}{N-mp}$ verifies (4.15), and for

$$\lambda < e^{-B(\infty)}\frac{\alpha(N-p)-N}{N^{1-p}\alpha^p(p-1)^{p-1}} = e^{-B(\infty)}N\frac{(N-mp)^{p-1}(m-1)}{m^p(p-1)^{p-1}} = \lambda(m),$$

we get

$$\left\||u_n|^{p-1}\right\|_{\frac{Nm}{N-mp},\sigma} \le C\|f_n\|_{m,\sigma},$$

for some constant C. Being $|f_n| \le |f|$, we get the thesis. □

Remark 4.1. We observe that, in particular, the result obtained in Theorem 4.2 provides estimates in terms of suitable Lebesgue norms of u_n, and f. Indeed, choosing $\sigma = \frac{Nm}{N-mp}$ in (4.12), and supposing that $\frac{Nm}{N-mp} \ge \max\{\frac{1}{p-1}, 1\}$, being $L^m(\Omega) \subset L\left(m, \frac{Nm}{N-mp}\right)$, if $\lambda < \lambda(m)$ we have that

$$\left\||u_n|^{p-1}\right\|_{\frac{Nm}{N-mp}} \le C\|f\|_m. \tag{4.18}$$

Clearly, if $p \ge 2$ no further assumption on $m \in \left]1, \frac{N}{p}\right[$ is needed to get (4.18). Otherwise, we have to require that $m \ge \frac{(p^*)'}{p}$. This additional hypothesis is due only to technical reasons, but, when $\lambda < \lambda(m)$, the estimate (4.18) holds also in the case $1 < m < \frac{(p^*)'}{p}$. For sake of completeness, we sketch the proof of (4.18) in the general case. We use the same notation of Theorem 4.2.

Let $\varepsilon > 0$, and $\alpha > 0$. By (4.10) we have:

$$-\frac{d}{dt}\int_{\{|u_n|>t\}}\frac{H(Du_n)^p}{(\varepsilon+|u_n|)^\alpha}dx \le e^{B(\infty)}(1+t)^{-\alpha}\int_0^{\mu_{u_n}(t)}[(c^+)^*u_n^*(\tau)^{p-1}+f^*(\tau)]\,d\tau,$$

and

$$\int_\Omega\frac{H(Du_n)^p}{(\varepsilon+|u_n|)^\alpha}dx \le e^{B(\infty)}\frac{1}{1-\alpha}\int_0^{|\Omega|}[(\varepsilon+u_n^*(s))^{1-\alpha}-\varepsilon^{1-\alpha}][(c^+)^*u_n^*(s)^{p-1}+f^*(s)]ds. \tag{4.19}$$

Now we recall that for any $\varepsilon > 0$ sufficiently small and $0 < \gamma < 1$, the following inequality holds:

$$x^{p-1}[(\varepsilon+x)^{p\gamma-(p-1)}-\varepsilon^{p\gamma-(p-1)}] \le [(\varepsilon+x)^\gamma-\varepsilon^\gamma]^p, \quad \forall x \ge 0.$$

Then we have, for $0 < \alpha < 1$,

$$\int_0^{|\Omega|} (c^+)^* u_n^*(s)^{p-1}[(\varepsilon + u_n^*(s))^{1-\alpha} - \varepsilon^{1-\alpha}]ds \le \tag{4.20}$$

$$\le \lambda \kappa_N^{\frac{p}{N}} \int_0^{|\Omega|} \frac{[(\varepsilon + u_n^*(s))^{1-\frac{\alpha}{p}} - \varepsilon^{1-\frac{\alpha}{p}}]^p}{s^{\frac{p}{N}}} ds = \lambda \kappa_N^{\frac{p}{N}} \|(\varepsilon + |u_n|)^{1-\frac{\alpha}{p}} - \varepsilon^{1-\frac{\alpha}{p}}\|_{p^*,p}^p \, .$$

Moreover,

$$\int_0^{|\Omega|} [(\varepsilon + u_n^*(s))^{1-\alpha} - \varepsilon^{1-\alpha}] f^*(s)\, ds \le C\|f\|_m \||u_n|^{1-\alpha}\|_{m'}. \tag{4.21}$$

As matter of fact, we have that by Hardy inequality (2.8),

$$\int_\Omega \frac{H(Du_n)^p}{(\varepsilon + |u_n|)^\alpha} dx = \left(\frac{p}{p-\alpha}\right)^p \int_\Omega H(D((\varepsilon+|u_n|)^{1-\frac{\alpha}{p}}))^p dx \ge \left(\frac{N-p}{p-\alpha}\right)^p \|((\varepsilon+|u_n|)^{1-\frac{\alpha}{p}} - \varepsilon^{1-\frac{\alpha}{p}})\|_{p^*,p}^p.$$

Using the above inequality, (4.20) and (4.21) in (4.19) we have that, by the properties of rearrangements and the Fatou lemma,

$$\left(\left(\frac{N-p}{p-\alpha}\right)^p - \frac{\lambda}{1-\alpha} e^{B(\infty)}\right) \|u_n^{1-\frac{\alpha}{p}}\|_{p^*,p}^p \le C\|f\|_m \|u_n^{1-\alpha}\|_{m'}.$$

Let choose α such that $(1-\alpha)m' = (1-\alpha/p)p^*$, after some computations we get that, being $m < \frac{(p^*)'}{p}$, then $0 < \alpha < 1$ and

$$(\lambda(m) - \lambda) \||u_n|^{p-1}\|_{\frac{Nm}{N-mp}} \le C\,(\|f\|_m)\,,$$

and for $\lambda < \lambda(m)$ we get the estimate (4.18).

Finally, the above estimate gives also a uniform bound for Du_n, that is

$$\|H(Du_n)^{p-1}\|_{\frac{Nm}{N-m}} \le C(\|f\|_m). \tag{4.22}$$

Clearly, if $m > \max\{1, \bar{m}\}$, $\bar{m} = \frac{N}{N(p-1)+1}$, this follows from (4.18) by Sobolev inequality. Otherwise, the above computations give that for $\varepsilon > 0$

$$\int_\Omega \frac{H(Du_n)^p}{(\varepsilon + |u_n|)^\alpha} dx \le C\|f\|_m \|(\varepsilon + |u_n|)^{1-\alpha}\|_{m'}.$$

Hence, reasoning as in [31], Hölder and Sobolev inequalities give (4.22).

Remark 4.2. We explicitly observe that if $m = (p^*)' = \frac{Np}{Np-N+p}$, then

$$\lambda(m) = \left(\frac{N-p}{p}\right)^p e^{-B(\infty)} = \Lambda_N e^{-B(\infty)}.$$

Next proposition will be an useful tool to pass to the limit in the approximating problems (4.11), and is a consequence of the obtained estimates on u_n.

Proposition 4.1. *Under the hypothesis of Theorem 4.2, for any $t > 0$ it holds that*

$$\int_0^{|\Omega|} [(c^+)^*(s) u_n^*(s)^{p-1} + f_n^*(s)]ds \le C\|f\|_{m,\sigma}, \tag{4.23}$$

and

$$\int_{\{|u_n|>t\}} |b_n(x, u_n, Du_n)| dx \le C t^{-\alpha}, \tag{4.24}$$

where $\alpha = \frac{(p-1)[N(m-1)+m(p-q)]}{N-mp} > 0$, and C denotes a positive constant independent of n and t.

Proof. The estimate (4.23) follows immediately from (4.12) and the definition of Lorentz space.

In order to show (4.24), let $t > 0$. Reasoning as in Theorem 4.1, and using the same notation, we have that

$$\begin{aligned}
(4.25)\qquad \int_{\{|u_n|>t\}} |b_n(x,u_n,Du_n)|dx &\le \int_{\{|u_n|>t\}} \beta(|u_n|)H(Du_n)^q dx \le \\
&\le \int_t^{+\infty} \beta(s)\left[\left(-\frac{d}{ds}\int_{\{|u_n|>s\}} H(Du_n)^p dx\right)^{\frac{q}{p}} (-\mu'_{u_n}(s))^{1-\frac{q}{p}}\right] ds \le \\
\le C\int_t^{+\infty} \beta(s)\left(\int_0^{\mu_{u_n}(s)} \left[(c^+)^*(\tau)u_n^*(\tau)^{p-1} + f^*(\tau)\right] d\tau\right)&(\mu_{u_n}(s))^{-\frac{p-q}{N'}} (-\mu'_{u_n}(s))^{p-q} ds,
\end{aligned}$$

where last inequality follows from (4.1) and (4.2). We always denote with C a constant independent of n. As matter of fact, the properties of Lorentz spaces give that $f \in L(m,\infty)$ and, by the estimate (4.12), $|u_n|^{p-1}$ are uniformly bounded in $L\left(\frac{mN}{N-mp},\infty\right)$. This implies that, for $s \ge t$,

$$\begin{aligned}
(4.26)\qquad \int_0^{\mu_{u_n}(s)} \left[(c^+)^*(\tau)u_n^*(\tau)^{p-1} + f^*(\tau)\right] d\tau &\le C\int_0^{\mu_{u_n}(t)} \left[\tau^{-\frac{p}{N}}\tau^{-\frac{N-mp}{Nm}} + \tau^{-\frac{1}{m}}\right] d\tau = \\
&= C\mu_{u_n}(t)^{1-\frac{1}{m}} \le Ct^{-\frac{N(p-1)(m-1)}{N-mp}}.
\end{aligned}$$

Hence, applying (4.26) in (4.25), we get

$$\int_{\{|u_n|>t\}} |b_n(x,u_n,Du_n)|dx \le Ct^{-\frac{N(p-1)(m-1)}{N-mp}} \int_t^{\infty} \beta(s)\,(\mu_{u_n}(s))^{-\frac{p-q}{N'}} (-\mu'_{u_n}(s))^{p-q} ds.$$

Hence, if $q = p$, the thesis follows immediately by (3.8). Otherwise, using the Hölder inequality, the hypothesis (3.8) and again the boundedness of u_n^{p-1} in $L(\frac{Nm}{N-mp},\infty)$, we get that

$$\begin{aligned}
\int_{\{|u_n|>t\}} |b_n(x,u_n,Du_n)|dx &\le Ct^{-\frac{N(p-1)(m-1)}{N-mp}} \left(\int_t^{+\infty} \beta(s)^{\frac{1}{q-(p-1)}} ds\right)^{q-(p-1)} \left(\int_0^{\mu_{u_n}(t)} s^{-\frac{1}{N'}} ds\right)^{p-q} \\
&\le Ct^{-\frac{N(p-1)(m-1)}{N-mp}} \mu_{u_n}(t)^{\frac{p-q}{N}} \le Ct^{-\alpha},
\end{aligned}$$

with $\alpha = \frac{(p-1)[N(m-1)+m(p-q)]}{N-mp} > 0$, and the proposition is completely proved. □

Now we consider the case $f \in L(m,\sigma)$, with $1 < m < (p^*)'$ and $1 < \sigma < +\infty$, and get some estimates for the Lorentz norm of the gradient of u_n.

Theorem 4.3. *Suppose that the hypotheses* (3.2) ÷ (3.7), (3.8) *hold. Let* $f \in L(m,\sigma)$*, with* $1 < m < (p^*)'$, $p' \le \sigma \le +\infty$*, and* $0 \le \lambda < \lambda(m)$*, with* $\lambda(m)$ *as in* (3.9)*. Then, the weak solutions* u_n *of* (4.11) *are such that*

$$(4.27)\qquad \left\|H(Du_n)^{p-1}\right\|_{\frac{Nm}{N-m},\sigma} \le C\|f\|_{m,\sigma},$$

for some constant C independent of n.

Proof. We reason similarly to the proof of Theorem 4.2. First of all, let $\sigma = +\infty$. Then, recalling (4.12) and that $(c^+)^*(s) \le \lambda\kappa_N^{\frac{p}{N}} s^{-\frac{p}{N}}$, we have

$$(c^+)^*(s)u_n^*(s)^{p-1} \le Cs^{-\frac{1}{m}}.$$

Hence, substituting in (4.3), and integrating, we get that

$$[H(Du_n)^*(s)]^{p-1} \leq Cs^{-\frac{N-m}{Nm}},$$

that gives (4.27) when $\sigma = +\infty$.

In the case $\sigma < +\infty$, by (4.3) we have:

$$\begin{aligned}\left\|H(Du_n)^{p-1}\right\|^{\sigma}_{d,\sigma} &= \int_0^{|\Omega|} \left(s^{\frac{1}{d}}[H(Du_n)^*(s)]^{p-1}\right)^{\sigma} \frac{ds}{s} \leq \\ &C\int_0^{|\Omega|}\left[s^{\frac{p'}{d}-\alpha-1}\int_0^s r^{\alpha-\frac{p'}{N'}}\left(\int_0^r z(t)dt\right)^{p'}dr\right]^{\frac{\sigma}{p'}}\frac{ds}{s}+ \\ &+C\int_0^{|\Omega|}\left[s^{\frac{p'}{d}-1}\int_s^{|\Omega|} r^{-\frac{p'}{N'}}\left(\int_0^r z(t)dt\right)^{p'}dr\right]^{\frac{\sigma}{p'}}\frac{ds}{s},\end{aligned}$$

with $z = (c^+)^*(u_n^*)^{p-1} + f_n^*$. Being $\alpha > \frac{p'}{N'} - 1$, if $N' < d < p'$, using the inequalities (2.6) and (2.7), we obtain that

$$\begin{aligned}\left\|H(Du_n)^{p-1}\right\|^{\sigma}_{d,\sigma} &\leq C\int_0^{|\Omega|}\left(s^{\frac{1}{d}+\frac{1}{N}}z(s)\right)^{\sigma}\frac{ds}{s} \leq \\ &\leq C\int_0^{|\Omega|}\left(s^{\frac{1}{d}+\frac{1}{N}-\frac{p}{N}}(u_n^*(s))^{p-1}\right)^{\sigma}\frac{ds}{s} + C\int_0^{|\Omega|}\left(s^{\frac{1}{d}+\frac{1}{N}}f_n^*(s)\right)^{\sigma}\frac{ds}{s}.\end{aligned}$$

Choosing d such that $\frac{1}{d}+\frac{1}{N}=\frac{1}{m}$, we have that $d=\frac{Nm}{N-m}$, and $N' < d < p'$, being $1 < m < (p^*)'$.

Being $\lambda < \lambda(m)$, we can use the estimates of Theorem 4.2, obtaining the thesis. □

5. Proofs of the existence and regularity theorems

Now we can prove the existence and regularity results for problem (3.1) stated in Section 3. Using the estimates of the previous section, we will pass to the limit in the approximating problems

$$\begin{cases} -\operatorname{div} a(x,u_n,Du_n) = b_n(x,u_n,Du_n) + g_n(x,u_n) + f_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where, as in the previous section, b_n, g_n and f_n are the truncates of b, g and f, respectively.

Proof of Theorem 3.1. As usual, we show that the solutions $u_n \in W_0^{1,p}(\Omega)$ to problem (5.1) found in Theorem 4.2 converge to a weak solution of (3.1), i.e. for any $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ it is possible to pass to the limit in

$$\int_\Omega a(x,u_n,Du_n)\cdot D\varphi\,dx = \int_\Omega [b_n(x,u_n,Du_n)+g_n(x,u_n)+f_n]\varphi\,dx. \tag{5.2}$$

By Theorem 4.1, being u_n bounded,

$$
\begin{aligned}
e^{-B(\infty)}\int_\Omega H(Du_n)^p dx &\le \int_0^\infty \left(\int_0^{\mu_{u_n}(t)} [(c^+)^*(s)u_n^*(s)^{p-1} + f^*(s)]\,ds\right)dt = \\
&= \int_0^{|\Omega|} (-u_n^*(r))' \left(\int_0^r [(c^+)^*(s)u_n^*(s)^{p-1} + f^*(s)]\,ds\right)dr = \\
&= \lambda \int_{\Omega^\star} \frac{(u_n^\star(x))^p}{H^o(x)^p}dx + \int_0^{|\Omega|} f^*(s)u_n^*(s)ds.
\end{aligned}
$$

Then, by the Hardy inequality (2.8) and the Hölder inequality we get

$$
e^{-B(\infty)}\int_\Omega H(Du_n)^p dx \le \frac{\lambda}{\Lambda_N}\int_{\Omega^\star} H(Du_n^\star)^p dx + \|f\|_{(p^*)',p'}\|u_n\|_{p^*,p}.
$$

Hence, by the Pólya-Szegö inequality we get that

$$
\left(e^{-B(\infty)} - \frac{\lambda}{\Lambda_N}\right)\int_\Omega H(Du_n)^p dx \le \|f\|_{(p^*)',p'}\|u_n\|_{p^*,p}.
$$

Recalling the Remark 2.1, and being $\lambda < \Lambda_N e^{-B(\infty)}$, we get that u_n is uniformly bounded in $W_0^{1,p}(\Omega)$ and, up to a subsequence,

$$
\begin{array}{ccll}
u_n & \rightharpoonup & u & \text{weakly in } W_0^{1,p}(\Omega), \\
g_n(x,u_n) & \to & g(x,u) & \text{strongly in } L^1(\Omega).
\end{array}
$$

Moreover, $b_n(x,u_n,Du_n)$ is bounded in $L^1(\Omega)$. Indeed, being β continuous, using (4.24) of Proposition 4.1, we get that

$$
\begin{aligned}
\int_\Omega |b_n(x,u_n,Du_n)|dx &\le \int_{\{|u_n|\le k\}} \beta(u_n)H(Du_n)^q dx + \int_{\{|u_n|>k\}} \beta(u_n)H(Du_n)^q dx \\
&\le |\Omega|^{1-\frac{q}{p}}\left(\int_\Omega H(Du_n)^p dx\right)^{\frac{q}{p}} \max_{[0,k]}\beta + Ck^{-\alpha}.
\end{aligned}
$$

Hence, we can apply the compactness result of [14], obtaining that $Du_n \to Du$ a.e. in Ω. Now we prove the strong convergence of $b_n(x,u_n,Du_n)$ to $b(x,u,Du)$ in L^1. If $q = p$, this can be shown by a standard procedure (see for instance [31] and the references therein). Otherwise, we use the Vitali Theorem. The equiintegrability of $b_n(x,u_n,Du_n)$ follows observing that, similarly as before,

$$
\begin{aligned}
\int_E |T_n(b(x,u_n,Du_n))|dx &\le \int_{\{|u_n|\le k\}\cap E} \beta(u_n)H(Du_n)^q dx + \int_{\{|u_n|>k\}} \beta(u_n)H(Du_n)^q dx \\
&\le |E|^{1-q/p}\left(\int_\Omega H(Du_n)^p dx\right)^{\frac{q}{p}} \max_{[0,k]}\beta + Ck^{-\alpha} \le C\left(|E|^{1-\frac{q}{p}}\max_{[0,k]}\beta + k^{-\alpha}\right).
\end{aligned}
$$

Finally, we observe that, recalling (3.3), $a(x,u_n,Du_n)$ is bounded in $L^{p'}(\Omega)$, and then it weakly converges to $a(x,u,Du)$ in $(L^{p'}(\Omega))^N$. Being $u \in W_0^{1,p}(\Omega)$, we can pass to the limit in (5.2), and this concludes the proof of part (i).

Clearly, if $f \in L(m,\sigma)$, $(p^*)' < m < \frac{N}{p}$ and $\max\{1, \frac{1}{p-1}\} \le \sigma \le +\infty$, by Theorem 4.2 the obtained solution u verifies the estimate in (ii). □

Proof of Theorem 3.2. Let u_n be a solution of (5.1). Then, the estimates (4.12) and (4.27) hold. As matter of fact, for a.e. $t > 0$ we have that

$$-\frac{d}{dt}\int_{\{|u_n|>t\}} H(Du_n)^p dx = \frac{d}{dt}\int_{\{|u_n|\le t\}} H(Du_n)^p dx.$$

Hence, applying (4.1) to u_n, and integrating between 0 and k, by (4.23) we have that

$$\int_\Omega H(DT_k(u_n))^p dx \le Ck. \tag{5.3}$$

Hence, $T_k(u_n)$ are bounded in $W_0^{1,p}(\Omega)$ and, up to a subsequence, $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega)$. Moreover, $|u|^{p-1} \in L\left(\frac{Nm}{N-mp}, \sigma\right)$, and $g_n(x,u_n) \to g(x,u)$ strongly in $L^1(\Omega)$. Similarly as in the proof of Theorem 3.1, by (4.24) we get

$$\begin{aligned}\int_E |b_n(x,u_n,Du_n)|dx &\le |E|^{1-q/p}\left(\int_\Omega H(DT_k(u_n))^p dx\right)^{\frac{q}{p}} \max_{[0,k]}\beta + Ck^{-\alpha} \le \\ &\le C\left(|E|^{1-\frac{q}{p}}k^{\frac{q}{p}}\cdot\max_{[0,k]}\beta + k^{-\alpha}\right),\end{aligned} \tag{5.4}$$

where last inequality follows from (5.3). Then, (5.4) gives that $b_n(x,u_n,Du_n)$ is bounded in $L^1(\Omega)$, and by the compactness result contained in [13] (see also [18]), $Du_n \to Du$ a.e. in Ω (up to a subsequence). If $q < p$, the Vitali Theorem assures the strong convergence of $b_n(x,u_n,Du_n)$ to $b(x,u,Du)$ in $L^1(\Omega)$, and the strong convergence of $T_k(u_n)$ to $T_k(u)$ in $W_0^{1,p}(\Omega)$. Otherwise, for $q = p$ this can be shown in a standard way using a suitable exponential test function (see for instance [31], and the reference therein). Hence, we can pass to the limit in the right-hand side of

$$\int_\Omega a(x,u_n,Du_n)\cdot DT_k(u_n-\varphi)\,dx = \int_\Omega [b_n(x,u_n,Du_n) + g_n(x,u_n) + f_n]T_k(u_n-\varphi)\,dx, \tag{5.5}$$

for all $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$.

As regards the left-hand side of (5.5), a simple argument based on the Fatou Lemma (see [12]) allows to show that

$$\int_\Omega a(x,u,Du)\cdot DT_k(u-\varphi)dx \le \liminf_{n\to+\infty}\int_\Omega a(x,u_n,Du_n)\cdot DT_k(u_n-\varphi)\,dx.$$

Then u verifies (3.11), and this concludes the proof of the existence result.

Finally, by Theorem 4.3 the obtained solution u verifies (3.12). □

Remark 5.1. In order to get an existence and regularity result for $f \in L^m$, $1 < m < (p^*)'$, we can repeat line by line the above proof using the estimates contained in Remark 4.1.

References

[1] B. Abdellaoui, A. Dall'Aglio, and I. Peral. Some remarks on elliptic problems with critical growth on the gradient. *J. Differential Equations*, 222:21–62, 2006.

[2] B. Abdellaoui and I. Peral. The equation $-\Delta u - \lambda\frac{u}{|x|^2} = |Du|^p + cf(x)$: the optimal power. *Ann. Mat. Pura Appl. (5)*, VI:1–25, 2007.

[3] B. Abdellaoui, I. Peral, and A. Primo. Some elliptic problems with Hardy potential and critical growth in the gradient: non-resonance and blow-up results. *J. Differential Equations*, 239:386–416, 2007.

[4] A. Alvino. Sulla diseguaglianza di Sobolev in spazi di Lorentz. *Boll. Un. Mat. Ital. A (5)*, 14(1):148–156, 1977.

[5] A. Alvino, V. Ferone, P.-L. Lions, and G. Trombetti. Convex symmetrization and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 14(2):275–293, 1997.

[6] A. Alvino, V. Ferone, and G. Trombetti. Estimates for the gradient of solutions of nonlinear elliptic equations with L^1 data. *Ann. Mat. Pura Appl. (4)*, 178:129–142, 2000.

[7] M. Amar and G. Bellettini. A notion of total variation depending on a metric with discontinuous coefficients. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 11(1):91–133, 1994.

[8] M. Belloni, V. Ferone, and B. Kawohl. Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators. *Zeitschrift fur Angewandte Mathematik und Physik (ZAMP)*, 54(5):771–783, Sept. 2003.

[9] M. Belloni, B. Kawohl, and P. Juutinen. The $p-$Laplace eigenvalue problem as $p \to \infty$ in a Finsler metric. *J. Eur. Math. Soc. (JEMS)*, 8(1):123–138, 2006.

[10] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vázquez. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 22(2):241–273, 1995.

[11] C. Bennett and R. Sharpley. *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1988.

[12] L. Boccardo. Some nonlinear Dirichlet problems in L^1 involving lower order terms in divergence form. In *Progress in elliptic and parabolic partial differential equations (Capri, 1994)*, volume 350 of *Pitman Res. Notes Math. Ser.*, pages 43–57. Longman, Harlow, 1996.

[13] L. Boccardo and T. Gallouët. Nonlinear elliptic equation with right hand side measures. *Comm. Partial Differential Equations*, 17(3&4):641–655, 1992.

[14] L. Boccardo and F. Murat. Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. *Nonlinear Anal.*, 19(6):581–597, 1992.

[15] L. Boccardo, L. Orsina, and I. Peral. A remark on existence and optimal summability of solutions of elliptic problems involving Hardy potential. *Discrete And Continuous Dynamical Systems*, 16(3):513–523, 2006.

[16] L. Boccardo, S. Segura, and C. Trombetti. Bounded and unbounded solutions for a class of quasi-linear elliptic problems with a quadratic gradient term. *J. Math. Pures Appl.*, 9:919–940, 2001.

[17] A. Cianchi and P. Salani. Overdetermined anisotropic elliptic problems. *Math. Ann.*, 345(4):859–881, 2009.

[18] T. Del Vecchio. Nonlinear elliptic equations with measure data. *Potential Anal.*, 4:185–203, 1995.

[19] F. Della Pietra. Existence results for non-uniformly elliptic equations with general growth in the gradient. *Differential Integral Equations*, 21(9-10):821–836, 2008.

[20] F. Della Pietra and N. Gavitone. Anisotropic elliptic problems involving Hardy-type potentials. *Preprint*, 2012.

[21] F. Della Pietra and N. Gavitone. Symmetrization for Neumann Anisotropic Problems and Related Questions. *Advanced Nonlinear Stud.*, 12:219–235, 2012.

[22] V. Ferone and B. Kawohl. Remarks on a Finsler-Laplacian. *Proc. Am. Math. Soc.*, 137(1):247–253, 2009.

[23] V. Ferone and B. Messano. Comparison and existence results for classes of nonlinear elliptic equations with general growth in the gradient. *Adv. Nonlinear Stud.*, 7(1):31–46, 2007.

[24] V. Ferone and F. Murat. Quasilinear problems having quadratic growth in the gradient: an existence result when the source term is small. In *Équations aux dérivées partielles et applications*, pages 497–515. Gauthier-Villars, Éd. Sci. Méd. Elsevier, Paris, 1998.

[25] V. Ferone and F. Murat. Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small. *Nonlinear Anal.*, 42(7, Ser. A: Theory Methods):1309–1326, 2000.

[26] J. P. García Azorero and I. Peral. Hardy inequalities and some critical elliptic and parabolic problems. *J. Differential Equations*, 144(2):441–476, Apr. 1998.

[27] N. Grenon, F. Murat, and A. Porretta. Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms. *Comptes Rendus Mathematique*, 342(1):23–28, Jan. 2006.

[28] N. Grenon, F. Murat, and A. Porretta. A priori estimates and existence for elliptic equations with gradient dependent terms. *Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser.*, to appear, 2012.

[29] K. Hansson, V. G. Maz'ya, and I. E. Verbitsky. Criteria of solvability for multidimensional Riccati equations. *Ark. Mat.*, 37:87–120, 1999.

[30] B. Kawohl and M. Novaga. The p-Laplace eigenvalue problem as $p \to 1$ and Cheeger sets in a Finsler metric. *J. Convex. Anal.*, 15(3):623–634, 2008.

[31] T. Leonori, P. J. Martínez-Aparicio, and A. Primo. Nonlinear elliptic equations with Hardy potential and lower order term with natural growth. *Nonlinear Anal.*, 74:3556–3569, July 2011.

[32] J. Leray and J.-L. Lions. Quelques résultats de Višik sur les problèmes elliptiques nonlinéaires par les méthodes de Minty-Browder. *Bull. Soc. Math. France*, 93:97–107, 1965.

[33] C. Maderna, C. D. Pagani, and S. Salsa. Quasilinear elliptic equations with quadratic growth in the gradient. *J. Differential Equations*, 97(1):54–70, 1992.

[34] A. Porretta. Nonlinear equations with natural growth terms and measure data. *Proceedings of the 2002 Fez Conference on Partial Differential Equations, Electron. J. Diff. Eqns. Conf. 09*, 9:183–202 (electronic), 2002.

[35] A. Porretta and S. Segura de León. Nonlinear elliptic equations having a gradient term with natural growth. *Journal de Mathématiques Pures et Appliquées*, 85(3):465–492, Mar. 2006.

[36] S. Segura de León. Existence and uniqueness for L^1 data of some elliptic equations with natural growth. *Adv. Differential Equations*, 9:1377–1408, 2003.

[37] G. Talenti. Elliptic equations and rearrangements. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 3(4):697–718, 1976.

[38] G. Talenti. Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces. *Ann. Mat. Pura Appl. (4)*, 120:160–184, 1979.

[39] C. Trombetti. Non-Uniformly Elliptic Equations with Natural Growth in the Gradient. *Potential Analysis*, 0(x):391–404, 2003.

[40] J. Van Schaftingen. Anisotropic symmetrization. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(4):539–565, July 2006.

Francesco Della Pietra, Università degli studi del Molise, Dipartimento di Bioscienze e Territorio, Via Duca degli Abruzzi, 86039 Termoli (CB), Italia.

E-mail address: francesco.dellapietra@unimol.it

Nunzia Gavitone, Università degli studi di Napoli "Federico II", Dipartimento di Matematica e Applicazioni "R. Caccioppoli", 80126 Napoli, Italia.

E-mail address: nunzia.gavitone@unina.it